AN ALTERNATIVE TREATMENT OF TRANSCENDENTAL FUNCTIONS IN CALCULUS

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1 Introduction

College calculus curriculum reform has been the topic of many research efforts and has gained national interest. There are approximately 600,000 students per year who enroll in calculus in four-year colleges and universities of the United States. Approximately half of these students are enrolled in an engineering calculus course. Of these students, less than half pass the course with a D or higher (see [2] and [6]). Concern for this problem has spurned curriculum development in areas that increase emphasis on numerical methods and the use of technology. As of 1994, it was estimated that approximately 32% of all calculus students were enrolled in some type of reform calculus course (see [8]).

Technology has allowed curriculum developers to rethink topics in a traditional calculus class; there has been a change in emphasis of some topics while others have been eliminated. But more importantly, calculus reform has focused more on how the content is actually taught and on the transferability of skills to work in multiple disciplines.

The calculus curriculum must prepare students to be successful in a variety of programs. Successful mathematicians, engineers, scientists, and economists of the future depend on faculty who demand quality performance from students at this level of their education. Excellence does not simply materialize in upper-division courses or when these students begin work in the field (see [1]). We believe that in any introductory calculus class, the focus should be raising students' conceptual understanding.

While there are many topics upon which we could concentrate, in this article we chose to focus on the treatment of transcendental functions. Traditionally, exponential, logarithmic, and inverse trigonometric functions are taught *after* elementary integration and the Fundamental Theorem of

Calculus have been introduced. While this practice is logically sound, it does have several shortcomings. First, these topics are introduced near the end of the first course. As such, students do not have adequate time in the course to become familiar and comfortable with these topics. Second, because of this timing, any serious gaps in students' knowledge of algebra and/or trigonometry will remain essentially undetected until the end of the course. Finally, the pool of available elementary functions is substantially smaller if one excludes building blocks of exponential, logarithmic, and inverse trigonometric types from the examples and exercises presented earlier in the course. In an attempt to eliminate some of these shortcomings, we were compelled to initiate a change in the way these topics are treated in an introductory calculus class. This approach neither adds nor deletes traditional topics. Rather it is a reorganization of topics, bringing a very early introduction of transcendental functions, presenting them in such a way as to build on the students' prior knowledge, especially their knowledge of algebra and trigonometry. Our quest is to promote greater understanding of the traditional topics.

As a case has been made for the benefits of an early introduction of transcendental functions, one might ask, "Is there any loss associated with the early introduction of these concepts?" The only one which we believe could be argued is "rigor". However, most introductory courses lack rigor. It is more important that we consider the needs of the students, the need to focus on understanding and communicating mathematics so that they can, in turn, become independent thinkers and mathematical problem solvers.

One of the most difficult, yet most essential tasks in the teaching of calculus (or any other mathematics) is helping students to think. In helping students to think, one must try to link new knowledge to what the students already know and understand. The National Council of Teachers' of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (see [5]) lists "mathematical connections" as one of its four goals. These connections include connections between mathematical topics within a specific course, connections between mathematical courses, and connections to the real world. Using this approach, we attempt to help students make those connections and help them to develop a deeper understanding of the topics.

The ideas which we present have probably been in the minds of many mathematicians. Those who have thought to rearrange the topics may have certainly presented analogous arguments. Indeed, similar work is done in texts where transcendental functions make an early appearance (i.e. before the Fundamental Theorem of Calculus has been introduced) (see [7], [4], and [3]). However, we believe these concepts can be introduced even earlier, provided that the definition of the derivative, the derivatives of power functions, and the six trigonometric functions have been introduced. Our goal is to give a complete and comprehensive view of the treatment of transcendental functions, by allowing the students to develop some of the theory for themselves. Many of the following problems are multi-faceted. They are written to carefully guide students while still allowing them to make discoveries on their own. We believe they will open the door for classroom discussion and trust that they can be used in classes where collaborative work is a focus. They are written with the intent that students will learn to communicate mathematically and develop a deeper understanding of fundamental concepts. Explicit and detailed solutions follow each problem for the readers' convenience.

2 Derivatives

The derivatives of transcendental functions can be introduced shortly after the development of derivatives of all other elementary functions. The prerequisites for this introduction include the study of limits, the definition of the derivative, and a reasonably strong background in algebra and trigonometry. Notice that the rules for differentiation have not yet been examined. The following problems are meant to give a compelling argument for the derivatives of the exponential, logarithmic, and inverse trigonometric functions.

We start with the inequality given below. It can be verified graphically or numerically.

Problem 2.1 (The derivatives of e^x and $\ln x$) *It can be shown that for all $x > -1$ we have*

$$e^{\frac{x}{x+1}} - 1 \leq x \leq e^x - 1$$

1. *Using this fact, compute* $\lim_{x\to 0} \dfrac{\ln(x+1)}{x}$
2. *Notice that $x \to 0$ if and only if $e^x - 1 \to 0$. So substitute $x = e^u - 1$ in part (1) and now with a bit of extra work, compute* $\lim_{u\to 0} \dfrac{e^u - 1}{u}$.
3. *Use parts 1 and 2 to find the derivatives of $f(x) = e^x$ and $g(x) = \ln x$.*

Solution:

1. From the inequality above we have that:

$$e^{\frac{x}{x+1}} \leq x + 1 \leq e^{x}$$

and so

$$(*) \qquad \frac{x}{x+1} \leq \ln(x+1) \leq x\,.$$

Notice that for $x > 0$, (*) yields

$$\frac{1}{x+1} \leq \frac{\ln(x+1)}{x} \leq 1$$

and so by the Squeeze Theorem we have that

$$\lim_{x\to 0^{+}} \frac{\ln(x+1)}{x} = 1\,.$$

Now for $x < 0$, (*) yields

$$1 \leq \frac{\ln(x+1)}{x} \leq \frac{1}{x+1}$$

and so by the Squeeze Theorem again, we have that

$$\lim_{x\to 0^{-}} \frac{\ln(x+1)}{x} = 1\,.$$

Thus,

$$\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1\,.$$

2. Let $x = e^{u} - 1$. If $u \to 0$, then $x \to 0$. Thus,

$$1 = \lim_{x\to 0} \frac{\ln(x+1)}{x} = \lim_{u\to 0} \frac{u}{e^{u} - 1}$$

and so $\lim_{u\to 0} \dfrac{e^{u} - 1}{u} = 1$ as well.

3. By the definition of the derivative we have

$$\begin{aligned}
\frac{d}{dx} e^x &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x (e^h - 1)}{h} \\
&= e^x \cdot \lim_{h\to 0} \frac{(e^h - 1)}{h} \\
&= e^x \cdot 1 \\
&= e^x
\end{aligned}$$

For $x > 0$, we also have by the definition of the derivative

$$\begin{aligned}
\frac{d}{dx} \ln x &= \lim_{h\to 0} \frac{\ln(x + h) - \ln(x)}{h} \\
&= \lim_{h\to 0} \frac{\ln\left(\frac{x + h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{\ln\left(1 + \frac{h}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{1}{x} \cdot \frac{\ln\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \\
&= \frac{1}{x} \cdot \lim_{t\to 0} \frac{\ln\left(t + 1\right)}{t} \\
&= \frac{1}{x} \cdot 1 \\
&= \frac{1}{x}
\end{aligned}$$

Problem 2.2 (Extensions to general exponential and logarithmic functions.) *Let r be a positive real constant with $r \neq 1$.*

1. *Use the change of base formula* (*that is* $\log_r(x) = \dfrac{\ln x}{\ln r}$) *and the value of the limit in part 1 of problem 2.1 to compute* $\lim_{x\to 0} \dfrac{\log_r(x+1)}{x}$.
2. *Keeping in mind the methods in part 2 of problem 2.1, compute* $\lim_{u\to 0} \dfrac{r^u - 1}{u}$.
3. *Find the derivatives of* $f(x) = r^x$ *and* $g(x) = \log_r x$.

Solution:

1. We have

$$\begin{aligned}
\lim_{x\to 0} \frac{\log_r(x+1)}{x} &= \lim_{x\to 0} \frac{\frac{\ln(x+1)}{\ln r}}{x} \\
&= \frac{1}{\ln r} \cdot \lim_{x\to 0} \frac{\ln(x+1)}{x} \\
&= \frac{1}{\ln r} \cdot 1 \\
&= \frac{1}{\ln r}
\end{aligned}$$

2. Set $x = r^u - 1$. Then $x \to 0$ if and only if $u \to 0$. Now,

$$\begin{aligned}
\frac{1}{\ln r} &= \lim_{x\to 0} \frac{\log_r(x+1)}{x} \\
&= \lim_{x\to 0} \frac{\log_r(r^u)}{r^u - 1} \\
&= \lim_{x\to 0} \frac{u}{r^u - 1}
\end{aligned}$$

So $\lim_{u\to 0} \dfrac{r^u - 1}{u} = \ln r$.

3. An argument identical to that of part 3 of problem 2.1 establishes that $\frac{d}{dx} r^x = (\ln r) \cdot r^x$ and $\frac{d}{dx} \log_r x = \frac{1}{x \ln r}$.

It is important to note that these problems provide students with a review of some of the more neglected ideas in algebra. These ideas include inverse functions, (specifically the relationship between exponential and logarithmic functions), the properties of exponents and logarithms, and the change of base formula. Problems 2.3 and 2.4 provide another approach to the same subject. Some may argue that these problems should be done first. We believe either approach is appropriate and the order of presentation depends more on the class make-up and the preference of the individual instructor.

Problem 2.3 (More extensions to general exponential functions.) *Suppose that f is a function such that*

$$f(a+b) = f(a) f(b) \quad \text{for all real numbers } a \text{ and } b.$$

Furthermore, suppose that

$$\lim_{x \to 0} \frac{f(x) - 1}{x} = c \quad \text{for some real number } c.$$

Show that f is differentiable with $f'(x) = cf(x)$ for all x. Can you think of any examples of functions of this type?

Solution:

We will show that f is differentiable by explicitly computing its derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= cf(x) \end{aligned}$$

Problem 2.4 (More extensions to general logarithmic functions.) *Suppose that f is a function so that*

$$f(a \cdot b) = f(a) + f(b) \quad \text{for all positive real numbers } a \text{ and } b.$$

Furthermore, suppose that

$$\lim_{x \to 0} \frac{f(x+1)}{x} = c \quad \text{for some real number } c.$$

Show that f is differentiable with $f'(x) = \frac{c}{x}$ *for all* $x > 0$. *Can you think of any examples of functions of this type?*

Solution:

Again we will show that f is differentiable by explicitly computing its derivative. So for $x > 0$ we have

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f\left(x\left(1 + \frac{h}{x}\right)\right) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x) + f\left(1 + \frac{h}{x}\right) - f(x)}{h} \\
&= \frac{1}{x} \cdot \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \\
&= \frac{1}{x} \cdot \lim_{t \to 0} \frac{f(1+t)}{t} \\
&= \frac{c}{x}
\end{aligned}$$

In most calculus texts, there is a section devoted to the topic of inverse functions. Here, the development of the derivative of inverse functions is presented. The theorem on the derivative of inverse functions is typically proven using the chain rule. Problem 2.5 draws upon the geometric interpretation of inverse functions as well as students' knowledge of algebra to illustrate this theorem. Allowing students to continue making connections with previously held knowledge, Problem 2.5 sets the stage for finding derivatives of inverse trigonometric functions.

Problem 2.5 (The derivative of the inverse function.) *Suppose that f is differentiable and one-to-one. Let L be the linear function that describes the tangent line on the graph of f at the point $(a, f(a))$. Assuming that L has a non-zero slope, then L is itself a one-to-one function.*

Furthermore, L^{-1} is nothing more than the tangent line on the graph f^{-1} at the point $(f(a),a)$.

1. Use this observation to deduce that

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

(Hint: If $L(x) = y = mx + b$ is any line with slope $m \neq 0$, find its inverse. What is the slope of the inverse?)

2. Let $x = f(a)$ in part 1 of the problem and find a formula for $(f^{-1})'(x)$.

Solution:

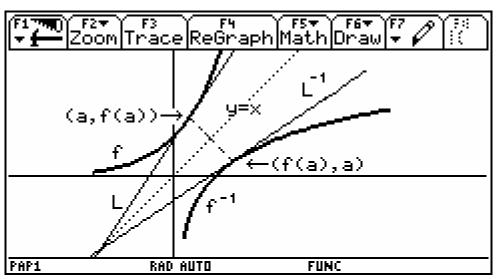

Figure 1

Notice that by using the geometric interpretation, students can literally see the derivative of the inverse function, connecting it to previous knowledge. (Figure 1)

1. Let $L(x) = y = mx + b$ be the tangent line on the graph of f at the point $(a, f(a))$. Then, $m = f'(a)$ and as long as $m \neq 0$, we compute $L^{-1}(x)$ by solving the equation $x = my + b$ for y.

Indeed $L^{-1}(x) = y = \frac{1}{m}x - \frac{b}{m}$ and so the slope of L^{-1} is $\frac{1}{m} = \frac{1}{f'(a)}$. Since L^{-1} is the tangent line on the graph of f^{-1} at the point ($f(a)$, a), we conclude that its slope $\frac{1}{f'(a)} = (f^{-1})'(f(a))$.

2. Recall that $x = f(a)$ if and only if $a = f^{-1}(x)$. So by part 1,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

This result is now used in the development of the derivative of inverse trigonometric functions.

Problem 2.6 *Use the result in problem 2.5 to find the derivatives of*

1. $f(x) = \tan^{-1} x$ in $(-\infty, \infty)$
2. $g(x) = \sin^{-1} x$ in (-1,1)
3. $h(x) = \sec^{-1} x$ in $(-\infty, -1) \cup (1, \infty)$

Solution:

1. For any real x we apply problem 2.5 to obtain

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{\sec^2(\tan^{-1} x)}.$$

By the use of the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$, we now have

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + \tan^2(\tan^{-1} x)} = \frac{1}{1 + x^2}.$$

2. Let x be in (−1, 1). Then thanks to problem 2.5 again we have

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)}.$$

Recall the fundamental Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$. Thus for any number θ, $\cos\theta = \pm\sqrt{1-\sin^2\theta}$. Since the cosine function is positive in the first and fourth quadrants, we have that for θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos\theta = \sqrt{1-\sin^2\theta}$. Keeping this in mind, the fact that $\sin^{-1} x$ is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, yields $\cos(\sin^{-1} x) = \sqrt{1-\sin^2(\sin^{-1} x)} = \sqrt{1-x^2}$. So

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

As you can see, both these derivatives were found without the use of the chain rule or implicit differentiation.

3. Let x be in $(-\infty, -1) \cup (1, \infty)$. By problem 2.5

$$\begin{aligned}\frac{d}{dx}\sec^{-1} x &= \frac{1}{\sec(\sec^{-1} x)\tan(\sec^{-1} x)} \\ &= \frac{1}{x\tan(\sec^{-1} x)}\end{aligned}$$

Recall again that $\sec^2\theta = 1 + \tan^2\theta$ and so $\tan\theta = \pm\sqrt{\sec^2\theta - 1}$. Hence,

$\tan(\sec^{-1} x) = \pm\sqrt{x^2-1}$. Now $\sec^{-1} x$ is in $(0, \frac{\pi}{2})$ for x in $(1, \infty)$ and $\sec^{-1} x$ is in $(\frac{\pi}{2}, \pi)$ for x in $(-\infty, -1)$, while the tangent function is positive on the first quadrant and negative on the second. Thus,

$$\tan(\sec^{-1} x) = \begin{cases} \sqrt{x^2-1} & \text{if} \quad x > 1 \\ -\sqrt{x^2-1} & \text{if} \quad x < -1 \end{cases}$$

So, for any x in $(-\infty, 1) \cup (1, \infty)$, $x\tan(\sec^{-1} x) = |x|\sqrt{x^2 - 1}$. Hence

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

It is often the case for students to wonder whether all "hard" limits, namely those that cannot be evaluated by straight substitutions, arise as derivatives. As we know, this may be true in some sense, L'Hospital's rule being the reason. It is perhaps noteworthy that a "utility grade" form of L'Hospital's rule is also within reach. That is, nothing but the definition of the derivative is needed to do the following problem.

Problem 2.7 (A primitive version of L'Hospital's rule.) *Suppose that f and g are differentiable at x = a with f(a) = g(a) = 0. Suppose that* $g'(a) \neq 0$. *Find* $\lim_{x\to a}\frac{f(x)}{g(x)}$.

Solution:

This is actually quite direct. Since f and g are at $x = a$ with $f(a) = g(a) = 0$, we have that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

$$= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

which by definition of the derivative together with the fact that $g'(a) \neq 0$, gives us $\dfrac{f'(a)}{g'(a)}$.

3 Rules of Differentiation

After the rules of differentiation have been discussed, the derivatives of the exponential and logarithmic functions can be revisited. These derivatives make a disguised appearance in the next two problems. We include this section mainly as reinforcement of crucial ideas for students.

Problem 3.1 (The derivatives of logarithms revisited.) *Suppose that f is differentiable on* $(0, \infty)$. *Assume that for all* $a > 0$, $b > 0$ *we have*

$$\textbf{(*)}\quad f(a \cdot b) = f(a) + f(b).$$

Furthermore, suppose that $f'(1) = c$ *for some constant c.*

1. *Use the chain rule to show that for any fixed number a we have* $\frac{d}{dx}f(ax) = af'(ax)$.
2. *Use (*) to show that* $\frac{d}{dx}f(ax) = f'(x)$.
3. *Find* $f'(a)$. *Can you think of examples of functions of this type?*

Solution:

1. This is already done. By the chain rule $\frac{d}{dx}f(ax) = af'(ax)$.
2. By (*) $\frac{d}{dx}f(ax) = \frac{d}{dx}(f(a) + f(x)) = \frac{d}{dx}f(x) = f'(x)$.
3. By parts 1 and 2, we have that $af'(ax) = f'(x)$ for all $x > 0$ and all $a > 0$. So for $x = 1$ we have $af'(a) = f'(1)$. Since $f'(1)$ is a constant, c, $f'(a) = \frac{c}{a}$.

Problem 3.2 (The derivatives of exponential functions revisited.) *Suppose that f is differentiable. Assume that for all numbers a and b we have*

$$\textbf{(*)}\ f(a+b) = f(a)\cdot f(b).$$

Furthermore, suppose that $f'(0) = c$ *for some constant c.*

1. *Use the chain rule to find* $\frac{d}{dx} f(x+a)$.
2. *Use* (*) *to show that* $\frac{d}{dx} f(x+a) = f(a)\cdot f'(x)$.
3. *Find* $f'(a)$. *Can you think of examples of functions of this type?*

Solution:

1. By a trivial application of the chain rule $\frac{d}{dx} f(x+a) = f'(x+a)$.
2. By (*) $\frac{d}{dx} f(a+x) = \frac{d}{dx}(f(a)\cdot f(x)) = f(a)\cdot \frac{d}{dx} f(x) = f(a)\cdot f'(x)$.
3. By parts 1 and 2, we have that $f'(x+a) = f(a)\cdot f'(x)$ for all real numbers x and a. So for $x = 0$ we have $f'(a) = f(a)\cdot f'(0)$. Since $f'(0)$ is a constant, c, $f'(a) = c\cdot f(a)$.

4 Conclusion

For many students, the topics of calculus are isolated and disjoint. For some, they are at best a set of procedures and routine practices. Such is the case when transcendental functions are isolated from others. A key to improving instruction is to help students make connections among the various topics.

A deeper understanding and appreciation of mathematics by students is the objective of any change in the mathematics curriculum. In this article, we have provided an alternative treatment of transcendental functions in introductory calculus. It is just that, an alternative method. While we believe it addresses some of the shortcomings that we find when transcendental functions are presented later in the course, only further research and assessment will enable us to determine whether it is a better treatment.

The aims of teaching mathematics need to include empowerment of learners to create their own mathematical knowledge. This can only be done by placing more emphasis on developing student understanding of concepts, building their problem solving and reasoning skills and helping them to make connections among concepts. We believe this method helps students to begin making these connections, by allowing them to build upon prior knowledge and use this knowledge to see how further concepts are developed.

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